Socially efficient discounting under ambiguity aversion\(^1\)

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Abstract

We consider an economy with an ambiguity-averse representative agent who faces an uncertain growth of her consumption. We show that it is not true in general that ambiguity aversion induces the representative agent to be more willing to save. In other words, ambiguity aversion does not necessarily reduce the equilibrium interest rate. We show that ambiguity aversion has a wealth effect and a pessimistic effect on the equilibrium interest rate. Under decreasing ambiguity aversion, ambiguity has an effect equivalent to reduce the expected future wealth, thereby reducing the interest rate. It has also the pessimistic effect to raise the probability of the worst scenarios, which generates in general an additional reduction of the interest rate.

Keywords: Discount rate, prudence, smooth ambiguity aversion, Ramsey rule, pessimism.
1 Introduction

The emergence of public policy problems associated with the sustainability of our economic growth has raised a considerable interest for the determination of a socially efficient discount rate. This debate has recently culminated with the publication of two reports. On one side, the Copenhagen Consensus (Lomborg (2004)) put top priority to public programs yielding immediate benefits (fighting malaria and AIDS, improving water supply,...), and rejected the idea to invest much in the prevention of global warming. On the other side, the Stern Review (Stern (2007)) put a tremendous pressure for acting quickly and heavily against global warming.

Because global warming will really affect our economies in more than 100 years, the choice of the rate at which these costs are discounted plays a key role in the conclusion. While Stern uses 1.4% per year, the Copenhagen Consensus applies a rate of 5% for costs and benefits of climate mitigation. For the sake of illustrating the power of discounting, consider a project which yields its benefits in \( t \) years time. For a horizon \( t = 100 \) the Copenhagen Consensus would require a rate-of-return already 36 times higher than Stern. This disagreement becomes even more pronounced as the horizon increases to \( t = 200 \), where critical benefits are 1300 times higher for the Copenhagen Consensus.

Yet, despite their disagreement on parameter values\(^1\), both aforementioned projects agree on the methods to determine the social discount rate. In particular, they consider an economy where the stochastic process which generates tomorrow’s wealth is perfectly known.

However, in view of the fact that typically predictions of economic growth within, say, 3 years time might already conflict, this assumption appears stark on the scale of decades, all the more when considering centuries. Several authors pointed out that structural parameter uncertainty might indeed be more important than risk for long time horizons. Weitzman (2007a) and Gollier (2007) show that as soon as one accepts uncertainty, the far distant future should be discounted at a smaller rate. Hence, whether the appropriate rate is closer to 1.4% than to 5%, not only depends on the preferences of the

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\(^1\)Most importantly, they disagree on the rate of pure preference for the present, i.e. how strongly, if at all, the well-being of future generations should be discounted. While the Copenhagen Consensus applies a rate of 1.5%, which is close to real world estimates of individuals, the Stern review advocates a rate of only 0.1%, based on ethical grounds. See Weitzman (2007b) for a discussion.
representative agent, but also on the model-uncertainty and the time horizon at hand.

Whereas these two papers are based on a completely standard expected utility modeling, we are motivated by the question, how to discount cash flows if agents dislike ambiguity. By that, we refer to a situation where the agent is confronted with a whole set of possible processes which might generate future wealth instead of only one. Following the pioneering work by Ellsberg (1961), ample evidence in favor of ambiguity aversion has been accrued\(^2\). It suggests that agents systematically violate Savage’s ”Sure Thing Principle”. More precisely, it seems that how we evaluate uncertainty depends on how precise our information about the underlying probabilities are - as opposed to the linearity of expected utilities in beliefs. Hence, a natural question to ask is, given ambiguity aversion, does a standard subjective utility model systematically under- (or over-) estimate the socially efficient discount rate?

We choose to address this question in allowing for “smooth ambiguity preferences” as proposed by Klibanoff, Marinacci and Mukerji (2005, 2007). First, unlike alternative specifications, this approach offers an intuitive separation between risk preferences and ambiguity preferences. This will be advantageous for our comparative statics analysis. Second, it relies on natural extensions of familiar concepts like certainty equivalents, which allows us to exploit tools from risk theory. In the subsequent sections we will identify joint requirements on tastes and uncertainty such that the discount rate is indeed lower if ambiguity preferences are taken into account.

We will first explain, why in general, ambiguity aversion may also increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utilities. On the one hand, an implicit *pessimistic effect* acts as if weights were shifted towards unfa-

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\(^2\)The *Ellsberg-Paradox* refers to the outcome of an experiment (Ellsberg (1961)), where a majority of participants’ choices were inconsistent with expected utility theory. In an urn containing 90 balls, there were 30 red balls and the remaining were either black or yellow in unknown proportions. If participants won a bet, they received $100. A large group preferred to bet on drawing *red* vs. betting on *black*. However, in a second stage they preferred to bet on *not* drawing *red* vs. betting on *not* drawing *black*. Within an expected utility model, one cannot find a set of beliefs compatible with such preferences. Note, that betting on (or against) *red* is indeed an unambiguous act with well-defined winning probabilities, while betting on (or against) *black* is not. For a survey on the literature consult e.g. Camerer and Weber (1992).
favorable distributions, while at the same time, an implicit wealth effect shifts the level of marginal expected utilities. We then go on to introduce the notion of non increasing absolute ambiguity aversion (DAAA), under which the wealth effect on savings is positive. Finally, we will derive pairs of conditions on the standard utility function and the stochastic ordering of candidate distributions which guarantee that indeed, under DAAA, the socially efficient discount rate is lower than in the subjective utility maximizer benchmark.

The effect of parameter uncertainty on discount rates has already been studied in standard subjective utility models. First and foremost, Weitzman (1998, 2001) and among others Groom et.al. (2004) examine the question, how misspecifications affect the efficient “certainty-equivalent” discount rate. Weitzman shows that asymptotically, for a large time horizon, the “certainty-equivalent” rate approaches the lowest among all plausible rates. While those models take rates as exogenously given, Weitzman (2007a) and Gollier (2007) determine endogenize them in a partial equilibrium model. Just like in the present paper, they also consider uncertainty on future economic growth. Similar to the present paper, those models find that equilibrium rates should be decreasing as we increase the time horizon.

Jouini et. al. (2007) consider the aggregation of diverging beliefs. They show that an aggregation bias might cause a richer evolution of the discount rate than in the representative agent models. In particular, the discount rate might be first increasing and only then approach its limit, namely the smallest individual rate. We believe that we can still defend our representative agent approach on the grounds of analytical convenience, given that we do not focus on the term structure of discount rates but on the impact of ambiguity aversion for a given time horizon.

Finally, our paper is technically related to Gollier (2006). In a finance context, he investigates comparative statics results of an increase in ambiguity aversion on the demand for risky assets. He shows that, in general, omitting ambiguity aversion cannot be corrected for by assuming a higher degree of risk aversion.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. Before deriving our main results in Section 5, we establish non increasing absolute ambiguity aversion. Section 6 investigates the effect
of an increase in ambiguity aversion and Section 7 concludes.

2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\tilde{c}_t$ fruits at date $t$, $t = 0, 1, 2, \ldots$. There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{rt}$ fruits for sure at date $t$. Thus, the real interest rate associated to maturity $t$ is $r_t$.

The distribution of $\tilde{c}_t$ is a function of a parameter $\theta$ that can take value $1, 2, \ldots, n$ with probability $q_1, \ldots, q_n$, respectively. The cumulative distribution function of $\tilde{c}_t$ conditional to $\theta$ is denoted $F_{t\theta}$. The random variable with such a cumulative distribution function is denoted $\tilde{c}_{t\theta}$. An ambiguous environment for $\tilde{c}_t$ is thus fully described by vector $C = (q_1, \tilde{c}_{t1}; \ldots; q_n, \tilde{c}_{tn})$.

Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji (2007)), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. For each plausible probability distribution $F_{\theta}$, the agent who purchased $\alpha$ zero-coupon bonds associated to date $t$ computes her future expected utility $U_t(\alpha, \theta) = \mathbb{E}_\theta[u(\tilde{c}_t + \alpha e^{rt})] = \int u(c + \alpha e^{rt})dF_{t\theta}(c)$ conditional to $F_{\theta}$ being the true distribution. We assume that $u$ is twice differentiable, increasing and concave, so that $U(\cdot, \theta)$ is concave in the investment $\alpha$, for all $\theta$. Ex ante, for a given investment $\alpha$, the welfare of the agent is measured by $V_t(\alpha)$ with

$$
\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_{\theta} \phi(U_t(\alpha, \theta)) = \sum_{\theta=1}^{n} q_{\theta} \phi \left( \mathbb{E}_\theta[u(\tilde{c}_t + \alpha e^{rt})] \right),
$$

The shape of $\phi$ describes the investor’s attitude towards ambiguity (or parameter uncertainty). Function $\phi$ is assumed to be three times differentiable, increasing and concave. $V_t(\alpha)$ can be interpreted as the certainty equivalent of the uncertain conditional expected utility $U_t(\alpha, \tilde{\theta})$. A linear $\phi$ means that the investor is neutral to ambiguity. In such a case, the DM is indifferent to any mean-preserving spread of $U_t(\alpha, \tilde{\theta})$, and $V_t(\alpha)$ can be represented by a subjective expected utility functional $V_t^{SEU}(\alpha) = \mathbb{E}_\theta[u(\tilde{c}_t + \alpha e^{rt})]$, where $\tilde{c}_t$ is the random variable that is distributed as $(\tilde{c}_{t1}, q_1; \ldots; \tilde{c}_{tn}, q_n)$. On the contrary, a concave $\phi$ is synonymous of ambiguity aversion in the sense that the DM dislikes any mean-preserving spread of the conditional expected utility $U_t(\alpha, \tilde{\theta})$. 4
An interesting particular case arises when the absolute ambiguity aversion
\[ A(U) = -\phi''(U)/\phi'(U) \]
is constant, so that \( \phi(U) = -A^{-1} \exp(-AU) \). As proved by Klibanoff, Marinacci and Mukerji (2005), the ex ante welfare \( V(\alpha) \) tends to maxmin expected utility functional
\[ V_t^{MEU}(\alpha) = \min_\theta E u(\tilde{c}_t + \alpha e^{rt}) \]
when the degree of absolute ambiguity aversion \( \phi \) tends to infinity. Thus, the Gilboa and Schmeidler (1989)’s maxmin criteria is a special case of this model.

The optimal investment \( \alpha^* \) maximizes the intertemporal welfare of the investor, which is written as:
\[ \alpha^* \in \arg \max_\alpha u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha). \quad (2) \]
Parameter \( \delta \) is the rate of pure preference for the present. If \( \phi \) and \( u \) are strictly concave, the objective function is concave in \( \alpha \) and the solution to program (2), when it exists, is unique. The necessary and sufficient condition of program (2) is written as
\[ u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*) \]
Fully differentiating equation (1) with respect to \( V \) yields
\[ V_t'(\alpha) = e^{rt} \sum_{\theta=1}^n q_{\theta} \phi' \left( E u(\tilde{c}_{t\theta} + \alpha e^{rt}) \right) \frac{E u'(\tilde{c}_{t\theta})}{\phi'(V_t(\alpha))} \]
Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity \( t \) is \( \alpha^* = 0 \). Combining the above two equations implies the following equilibrium condition:
\[ r_t = \delta - \frac{1}{t} \ln \left[ \frac{\sum_{\theta=1}^n q_{\theta} \phi' \left( E u(\tilde{c}_{t\theta}) \right) E u'(\tilde{c}_{t\theta})}{u'(c_0) \phi'(V_t(0))} \right]. \quad (3) \]
This is also the socially efficient rate at which sure benefits and costs occurring at date \( t \) must be discounted in any cost-benefit analyses at date 0.

As a benchmark, consider the case of an ambiguity neutral representative agent. In that case, we get the standard bond pricing formula
\[ r_t = \delta - t^{-1} \ln \left[ \frac{E u'(\tilde{c}_t)}{u'(c_0)} \right]. \]
In this special case, we see that the riskiness of future consumption reduces the socially efficient discount rate if and only if

\[ \text{See for example Cochrane (2001).} \]
\( u' \) is convex, i.e., if the representative agent is prudent (Leland (1968), Drèze and Modigliani (1972), Kimball (1990)).

Our goal in this paper is to determine the conditions under which ambiguity and ambiguity aversion reduce the discount rate. An ambiguous environment \( C = (q_1, \tilde{c}_t; \ldots ; q_n, \tilde{c}_tn) \) is said to be acceptable if the supports of the \( \tilde{c}_t \) are in the domain of \( u \), and if all \( Eu' (\tilde{c}_t) \) are in the domain of \( \phi \). The set of acceptable ambiguous environment is denoted \( \Psi \).

3 An analytical solution

Let us consider the following specification:

- The plausible distributions of \( \ln \tilde{c}_t \) are all normally distributed with the same variance \( \sigma^2 t \), and with mean \( \ln c_0 + \theta t \).
- The prior distribution on \( \theta \) is normally distributed with mean \( \mu \) and variance \( \sigma^2_0 \).
- The representative agent’s preferences exhibit constant relative risk aversion \( \gamma = -cu''(c)/u'(c) \), i.e., \( u(c) = c^{1-\gamma}/(1-\gamma) \).
- The representative agent’s preferences exhibit constant relative ambiguity aversion \( \eta = -|u| \phi''(u)/\phi'(u) \geq 0 \). This means that \( \phi(U) = h(kU)^{1-\eta/k} , \) where \( k = \text{sign}(1-\gamma) \) is the sign of \( u \).

We show in the Appendix that there is an analytical solution for the discount rate, which is given in this case by:

\[
 r_t = \delta + \gamma \mu - \frac{1}{2} \gamma^2(\sigma^2 + \sigma^2_0 t) - \frac{1}{2} \eta \frac{1 - \gamma^2}{\text{sign}(1-\gamma)} \sigma^2_0 t. \tag{4}
\]

The first two terms in the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the expected growth rate of consumption \( \mu \). When \( \mu \) is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in

\footnote{In continuous time, this would mean that the consumption process is a geometric brownian motion \( d\ln c = \theta dt + \sigma dw \).}

\footnote{We consider the natural continuous extension of our model with a discrete distribution for \( \theta \).}
the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\tilde{c}_t)$ under prudence, this has a negative impact on the discount rate. Notice that the variance of consumption at date $t$ equals $\sigma^2_t + \sigma^2_0 t^2$, so that its increases at an increasing rate with respect to the time horizon. There, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2007) to justify a decreasing discount rate in an expected utility framework.

The last term in the right-hand side of equation (4) characterizes the effect of ambiguity. Observe that it always tends to reduce the discount rate under positive ambiguity aversion ($\eta > 0$). This effect is increasing in the degree of ambiguity aversion $\eta$, in the degree of uncertainty $\sigma_0$, and in the time horizon $t$.

Note, that in our example, in the absence of ambiguity (i.e. $\sigma_0^2 = 0$), the term structure is flat. The effect of ambiguity is to decrease the rates linearly over time. Ambiguity aversion steepens this decline. More concretely, consider parameters taken from Weitzman (2007b) who proposes a "quartet of twos" (i.e. $\delta = 2\%$, $\gamma = 2$, $E[g] = 2\%$, $\sigma = 2\%$), which translates into an efficient discount rate of 5.9%. Then, if there is a small amount of ambiguity, described by $\sigma_0 = 0.5\%$, the socially efficient discount rate for a time horizon $t = 200$ falls by 1% to 4.9%. If in addition, there is ambiguity aversion with, say $\eta = 2$, the rate falls as low as 3.4%. Again, the leverage of this latter 1.5% difference comes through compounding it over 200 years – for a given future benefit the $\eta = 2$ society would be willing to give up 20 times more than the SEU society. And an increase in the ambiguity of the forecasts to a level of $\sigma_0 = 1\%$ amplifies this factor to 121.

In the remainder, we will investigate whether it is true in general, that ambiguity aversion decreases the socially efficient discount rate. Contrary to the example presented above, the next section finds that ambiguity aversion might even have the opposite effect, namely to increase the rate.

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6 An agent is prudent if adding risk to future incomes raises the precautionary saving (Kimball (1990)). Leland (1968) has shown that an agent is prudent if and only if her marginal utility is a convex function of consumption. Power utility functions exhibit prudence.

7 Under a normal distribution, this implies for instance that 95% of the scenarios lie in the range of $E[g] \in [1\%; 3\%]$. 

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4  The case of risk neutrality

In this section, we examine the case of risk neutrality, \( u(c) = c \). Let \( \bar{c}_\theta = E\bar{c}_t \) denote the expected consumption in date \( t \) conditional to \( \theta \). In that case, the pricing formula (3) is rewritten as

\[
    r_t = \delta - \frac{1}{t} \ln \left[ \frac{\sum_{\theta=1}^{n} q_\theta \phi'(\bar{c}_\theta)}{\phi'(V_t(0))} \right],
\]

with \( \phi(V_t(0)) = \Sigma_\theta q_\theta \phi(\bar{c}_\theta) \). Under ambiguity neutrality (\( \phi'' \equiv 0 \)), we would have \( r_t = \delta \). The following Lemma is useful for future results.

**Lemma 1** Consider a three times differentiable increasing function \( g: B \subset R \rightarrow R \). The following two conditions are equivalent:

1. For all random variables \( \tilde{x} \) whose support is in \( B \), and any scalar \( x_0 \),
   
   \[ E g(\tilde{x}) = g(x_0) \implies E g'(\tilde{x}) \geq g'(x_0). \]

2. \( g \) is such that \( -g''(x)/g'(x) \) is nonincreasing in \( x \).

**Proof:** Let us observe that condition 2 means that \( -g''/g' \) be uniformly larger than \( -g''/g' \), or that \( -g' \) is a concave transformation of \( g \). Thus, this means that there exists a concave function \( f \) such that \( -g'(x) = f(g(x)) \) for all \( x \).

Let us first prove that condition 1 holds for all random variables \( \tilde{x} \) if \( g \) satisfies condition 2. Indeed, because we have that \( -g' \) is a concave function \( f \) of \( g \), we have that

\[
    -E g'(\tilde{x}) = E f(g(\tilde{x})) \leq f(E g(\tilde{x})) = f(g(x_0)) = -g'(V)
\]

where we used Jensen’s inequality. Thus, we have proved \( 2 \implies 1 \).

To prove \( 1 \implies 2 \), let us assume by contradiction that \( -g' = f(g) \) is not a concave transformation of \( g \). This means that function \( f \) is locally convex in an interval \( [g(x_a), g(x_b)] \subset B \). Then take any distribution of \( \tilde{x} \) with a support in in \( [x_a, x_b] \). Then, it implies by the Jensen inequality that

\[
    -E g'(\tilde{x}) = E f(g(\tilde{x})) \geq f(E g(\tilde{x})) = f(g(x_0)) = -g'(V)
\]

This contradicts condition 1.\( \blacksquare \)

Applying this lemma for \( g = \phi \) and \( \tilde{x} \) being distributed as \( (q_1, \bar{c}_{t1}; \ldots; q_n, \bar{c}_{tn}) \) directly yields the following result.

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8See Gollier and Kimball (1996) and Gollier (2001, section 2.5)
Proposition 1 Suppose that the representative agent is risk neutral. The socially efficient discount rate is smaller than under ambiguity neutrality for all acceptable ambiguous environments \( C \in \Psi \) if and only if \( \phi \) exhibits nonincreasing absolute ambiguity aversion, i.e., iff 

\[
A'(U) = \left( -\phi''(U)/\phi'(U) \right)' \leq 0
\]

for all \( U \).

Under risk neutrality, the driving force for the impact of ambiguity on the interest rate is not ambiguity aversion itself, but whether the degree of ambiguity aversion is increasing or decreasing with the level of expected utility \( U \). In particular, Proposition 1 tells us that, under risk neutrality, the interest rate is always increased by ambiguity if ambiguity aversion is increasing with \( U \). In the limit case with risk neutrality and constant absolute ambiguity aversion, ambiguity has no effect on the equilibrium interest rate.

However, we now show that under risk aversion, non increasing ambiguity aversion (DAAA) is not enough to guarantee that ambiguity reduces the socially efficient discount rate. In our counterexample, \( c_0 = 2 \), and there are only \( n = 2 \) plausible distribution functions \( F_1 \) and \( F_2 \) for future consumption \( \tilde{c}_t \). The corresponding conditional distributions are depicted in Figure 1. We assume that the these two distributions are equally likely to be the true one, i.e., \( q_1 = q_2 = 1/2 \). We assume that the agent has a constant relative risk aversion \( \gamma = 2 \), i.e., \( u(c) = -c^{-1} \). We assume that the rate of pure preference for the present \( \delta \) equals zero. It is easy to check that the interest rate equals 9.24% in that economy if the representative agent would be neutral to ambiguity. Suppose alternatively that she has constant absolute ambiguity aversion (CAAA) with \( A = 2.11 \), i.e., \( \phi(U) = -\exp(-2.11U) \). Then, tedious computations lead to the conclusion that the socially efficient discount rate should be exactly zero in that economy: \( r_t = 0 \). Thus, this example demonstrates the fact that DAAA is not enough to guarantee that ambiguity about future consumption reduces the discount rate.

5 Sufficient conditions

In this section, we provide some sufficient conditions to guarantee that ambiguity reduces the equilibrium interest rate when the representative agent is risk-averse and has non increasing absolute ambiguity aversion. In case the
Figure 1: The two equally likely probability distributions of future consumption in the counter example

representative agent is neutral to ambiguity, the discount rate should equal

\[
\delta - \frac{1}{t} \ln \left[ \frac{Eu'(\tilde{c}_t)}{u'(c_0)} \right],
\]

where \( \tilde{c}_t \) describes future consumption, which is distributed as \((\tilde{c}_{t1}, q_1; \ldots; \tilde{c}_{tn}, q_n)\). Under ambiguity aversion, the pricing formula (3) can be rewritten as

\[
r_t = \delta - \frac{1}{t} \ln \left[ \frac{aEu'(\tilde{c}_t^\circ)}{u'(c_0)} \right],
\]

with

\[
a = \sum_{\theta=1}^{n} q_\theta \phi'(Eu(\tilde{c}_t^\circ))/\phi'(V_t(0))
\]

and where \( \tilde{c}_t^\circ \) is a distorted probability distribution \((\tilde{c}_{t1}, q_1^\circ; \ldots; \tilde{c}_{tn}, q_n^\circ)\) of future consumption, with

\[
q_\theta = \frac{q_\theta \phi'(Eu(\tilde{c}_t^\circ))}{\sum_{\tau=1}^{n} q_{\tau} \phi'(Eu(\tilde{c}_\tau^\circ))},
\]

for \( \theta = 1, \ldots, n \). Thus ambiguity aversion reduces the discount rate if

\[
aEu'(\tilde{c}_t^\circ) \geq Eu(\tilde{c}_t).
\]
We see from this analysis that ambiguity aversion has two effects on the discount rate. First, it has a wealth effect. To see this, define $b$ such that $aE\mu'(\tilde{c}_t) = E\mu'(\tilde{c}_t + b)$, with the property that $b$ is positive/negative when $a$ is smaller/larger than unity, where $a$ is defined by equation (8). If ambiguity aversion implies that $a$ is larger than unity, this would have a negative wealth effect ($b < 0$). As is well-known, a reduction in expected future wealth reduces the interest rate. From the definitions (1) and (8) of respectively $V_t(0)$ and $a$, it is immediate that $a$ is indeed larger than unity if $\phi$ exhibits decreasing absolute ambiguity aversion. Notice that $a$ equals unity when $\phi$ exhibit constant absolute ambiguity aversion, so that the wealth effect disappears in that case.

In addition to the wealth effect, there is a pessimistic effect. In the pricing formula (7), the expected marginal utility is computed by using the distorted random variable $\tilde{c}_t$ rather than the original $\tilde{c}_t$. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (9). The remainder of this section is devoted to characterize how the distortion affects the discount rate. Notice, that we would be done, once we find that $\tilde{c}_t$ is dominated by $\tilde{c}_t$ in the sense of first-degree stochastic dominance (FSD), because it would directly imply that $E\mu'(\tilde{c}_t)$ be larger than $E\mu'(\tilde{c}_t)$, since $u'$ is decreasing.

We will start by investigating how the the weight $q_\theta$ for different scenarios $\theta$ change. For that matter we will make use of the following concept.

**Definition 1** Two vectors $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are said to be **anti-comonotonic** if, for all $(i, j) \in \{1, n\}^2$, $x_i \leq x_j$ implies $y_i \geq y_j$.

Take the case where the entries $x_\theta$ of the vector are ranked and ordered such that they are decreasing in $\theta$. Such a vector is anti-comonotonic with any vector of the same dimension whose entries $y_\theta$ are increasing in $\theta$.

Consider now the distortion of beliefs $\tilde{q}_\phi$. It follows from the analysis above, that $\tilde{q}_\phi$ is dominated by $q$ in the sense of the monotone likelihood ratio order (MLR) if $\tilde{q}_\theta/q_\theta$ and $E\mu(\tilde{c}_\theta)$ are anti-comonotonic. Further, observe from (9), that $\tilde{q}_\theta/q_\theta$ is proportional to $\phi'(E\mu(\tilde{c}_\theta))$. Hence we can immediately state the following.

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9 $\tilde{x}$ dominates $\tilde{y}$ in the sense of FSD if $Eh(\tilde{x})$ is larger than $Eh(\tilde{y})$ for all functions $h$ that are non-decreasing. Second-degree stochastic dominance is a weaker condition because it restricts functions $h$ to be increasing and concave. The Rothschild-Stiglitz’s increases in risk correspond to the set of concave functions $h$. 

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Remark 1 Suppose without loss of generality that $Eu(\tilde{c}_{t1}) \leq \cdots \leq Eu(\tilde{c}_{tn})$. The following two conditions are equivalent:

1. Beliefs $\tilde{q}^\phi$ are dominated by $q$ in the sense of the monotone likelihood ratio order, for any set of marginals $(\tilde{c}_{t1}, \ldots, \tilde{c}_{tn})$ satisfying the above-mentioned ranking.

2. $\phi$ is concave.

This result has a very intuitive interpretation. Ambiguity aversion is characterized by an MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\tilde{c}_{t\theta}$ than marginal $\tilde{c}_{t\theta'}$, then, the ambiguity-averse representative agent increases the implicit prior probability $q_{t\theta'}$ relatively more than the implicit prior probability $q_{t\theta}$. This gives some flesh to our terminology in which we refer to a pessimistic effect for the distortion of implicit beliefs.

At this stage, we found two requirements on $\phi$ which guarantee a negative wealth effect and, for a given ranking of scenarios, a MLR-deterioration of beliefs. However, to ensure an unambiguous effect on the socially efficient discount rate, we have to consider how the implicit change in beliefs affect marginal expected utilities. Thus, we need to characterize the ranking of marginals $\tilde{c}_{t\theta}$ according to $u'$.

Lemma 2 Suppose that $\phi$ exhibits non increasing absolute ambiguity aversion, $(-\phi''(U)/\phi'(U))' \leq 0$ for all $U$. Then, ambiguity reduces the discount rate if $(Eu(\tilde{c}_{t\theta}))_{\theta=1,\ldots,n}$ and $(Eu'(\tilde{c}_{t\theta}))_{\theta=1,\ldots,n}$ are anti-comonotonic.

Proof: Because $\phi'$ is decreasing, we have that $(\phi'(Eu(\tilde{c}_{t\theta})))_{\theta=1,\ldots,n}$ and $(Eu'(\tilde{c}_{t\theta}))_{\theta=1,\ldots,n}$ are comonotonic. By the covariance rule, it implies that

$$\sum_{\theta=1}^n q_\theta \phi'(Eu(\tilde{c}_{t\theta})) Eu'(\tilde{c}_{t\theta}) \geq a \left[ \sum_{\theta=1}^n q_\theta Eu'(\tilde{c}_{t\theta}) \right].$$

By Lemma 1, we know that $a = \sum_{\theta=1}^n q_\theta \phi'(Eu(\tilde{c}_{t\theta})) / \phi'(V_t(0))$ is larger than unity under DAAA. This implies that the left-hand side of the above equality is larger than $\sum_{\theta} q_\theta Eu'(\tilde{c}_{t\theta}) = Eu'(\tilde{c}_t)$. This implies that the interest rate is reduced by ambiguity aversion.■
In other words, decreasing absolute ambiguity aversion would always reduce the interest rate if, for any pair \((\theta, \theta') \in \{1, \ldots, n\}^2\),

\[ Eu(\tilde{c}_\theta) \leq Eu(\tilde{c}_\theta') \quad \Rightarrow \quad -Eu'(\tilde{c}_\theta) \leq -Eu'(\tilde{c}_\theta'). \] (11)

Hence, given DAAA, it suffices to consider statistical properties of the marginals to ensure a clear-cut effect on the interest rate. The intuition comes from Remark 1, namely that the shift in beliefs constitutes an MLR-deterioration, which is a special case of a FSD-deterioration. As is well-known, for any function increasing in \(\theta\), the expectation decreases after a FSD-deterioration. Indeed, Lemma 2 tells us that it suffices to find out whether conditional marginal utilities \(-Eu(\tilde{c}_\theta)\) are decreasing in \(\theta\).

In other words, we need to identify the stochastic ordering such that both \(u\) and \(-u'\) agree on a ranking of scenarios \(\theta\). For that matter, we open a parenthesis on stochastic dominance orderings. In particular, apart from first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD), we will make use of another, weaker ordering.

**Definition 2** We say that \(\tilde{x}\) is riskier than \(\tilde{y}\) in the sense of Jewitt if the following condition is satisfied for all concave \(u\): If agent \(u\) prefers \(\tilde{y}\) to \(\tilde{x}\), then all agents more risk-averse than \(u\) also prefer \(\tilde{y}\) to \(\tilde{x}\).

The following definition states under which conditions distribution function \(F_{t\theta'}\) actually dominates \(F_{t\theta}\) in the sense of Jewitt (1989).

**Definition 3** Suppose that \(F_{t\theta}\) does not dominate \(F_{t\theta'}\) according to SSD. Then, for increasing and concave functions \(u\), \(F_{t\theta'}\) dominates \(F_{t\theta}\) in the sense of Jewitt, if and only if there exists some \(z\) in their support \([a, b]\), such that

\[ \int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \quad \text{for all} \quad x \in [a, z], \] (12)

\[ \int_a^z (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0 \] (13)

\[ \int_a^z (F_{t\theta'}(z) - F_{t\theta}(z))dz \quad \text{is nondecreasing on} \quad [z, b]. \] (14)

Two random variables fulfill Definition ?? if there exists point \(z\) in their support such that, conditional on the outcome being lower than \(z\), \(F_{t\theta'}\) dominates \(F_{t\theta}\) in the sense of SSD, whereas conditional on the outcome being
higher than \( z \), \( F_{t\theta'} \) is dominated by \( F_{t\theta} \) in the sense of FSD. Observe, that second-degree stochastic dominance is indeed stronger than Jewitt’s ordering, since SSD is contained in Definition ?? as a special case when we pick \( z = b \). Note, that unlike FSD or SSD, Jewitt’s ordering is compatible with a broader mean-variance trade-off, in the sense that \( \tilde{c}_{t\theta'} \) might be preferred to \( \tilde{c}_{t\theta} \) even though the latter has a higher mean.

Combining the above, with well-known properties of FSD and SSD enables us to state the following result.

**Lemma 3** \( \mathrm{Eu}(\tilde{c}_{t\theta})_{\theta=1,...,n} \) and \( (\mathrm{Eu}'(\tilde{c}_{t\theta}))_{\theta=1,...,n} \) are anti-comonotonic, if one of the following conditions hold,

1. The set of marginals \( (\tilde{c}_{t1},...\tilde{c}_{tn}) \) can be ranked according to FSD and \( u \) is increasing and concave.
2. The set of marginals \( (\tilde{c}_{t1},...\tilde{c}_{tn}) \) can be ranked according to SSD and \( u \) is increasing, concave and exhibits prudence.
3. The set of marginals \( (\tilde{c}_{t1},...\tilde{c}_{tn}) \) can be ranked according to Jewitt (1989) and \( u \) is increasing and concave and exhibits DARA.

**Proof:** To prove that condition 1. implies anti-comonotonicity, note that, by definition, \( \tilde{c}_{t\theta'} \) dominates \( \tilde{c}_{t\theta} \) in the sense of FSD, if for all increasing \( v \),

\[
\mathrm{Ev}(\tilde{c}_{t\theta'}) \geq \mathrm{Ev}(\tilde{c}_{t\theta}).
\]

Without loss of generality rank the marginals such that

\[
\theta' \geq \theta \Rightarrow \tilde{c}_{t\theta'} \succeq_{\text{FSD}} \tilde{c}_{t\theta}.
\]

The fact that for an increasing and concave function \( u \), both \( u \) and \( -u' \) are increasing yields the result.

In order to prove that condition 2. yields anti-comonotonicity, proceed analogously, exploiting that both \( u \) and \( -u' \) are increasing and concave. And similarly, 3. can be proved by exploiting that DARA implies that \( -u' \) is a concave transformation of \( u \). \( \blacksquare \)

Since, by assumption, our representative agent exhibits risk-aversion, the following set of Propositions combine the preceding results.
Proposition 2 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to first-degree stochastic dominance. Then, ambiguity aversion reduces the socially efficient discount rate.

The ranking of conditional distributions according to first-degree stochastic dominance is a relatively restrictive condition that would be desirable to relax. Second-degree stochastic dominance (SSD) is weaker than FSD, and it contains Rothschild-Stiglitz’s increases in risk as a particular case. However, part 2 of Lemma ?? indicates that we have to impose the convexity of $u'$ to obtain the following Proposition.

Proposition 3 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to second-degree stochastic dominance. Then, ambiguity aversion reduces (resp. increases) the socially efficient discount rate if $u'$ is convex (resp. concave).

Notice that in the counter-example described by Figure 1, the two random variables $\tilde{c}_1$ and $\tilde{c}_2$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

Proposition ??, requires that the coefficient of absolute prudence in $u$ is positive. The next natural step is to consider the family of utility functions which exhibit decreasing absolute risk-aversion (DARA). Obviously, with $u$ concave, the latter implies prudence, since it requires $-u'$ to be more concave than $u$.

Proposition 4 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to Jewitt. Then, ambiguity aversion reduces (resp. increases) the socially efficient discount rate if $u$ exhibits decreasing absolute risk-aversion (DARA).

Even the weakest ordering only requires what is considered to be a plausible property of risk-attitudes. The assumption of decreasing risk-aversion is widely accepted in the Economic literature and it is in particular compatible with the observation that more wealthy individuals tend to invest more in stocks.
6 Effects of an increase in ambiguity aversion

Above we characterized the effect of smooth ambiguity aversion on the willingness to save, starting from the subjective expected utility benchmark. The following section is devoted to more generally describe how the discount rate changes when ambiguity aversion increases.

For that matter, consider two economies \( i = 1, 2 \), identical up to their level of ambiguity aversion. In particular, suppose that the representative agent in economy 2 is more ambiguity-averse, such that \( \phi_2(U) = k(\phi_1(U)) \), with \( k(\cdot) \) increasing and concave. According to the adjusted pricing formula in (7) an increase in ambiguity aversion decreases the social discount rate if and only if

\[
a_2 E u'(\tilde{c}_{t2}) \geq a_1 E u'(\tilde{c}_{t1}),
\]

where \( a_i \) is defined like in (8) and where \( \tilde{c}_i \) is the implicit consumption distorted by weights \( q_\theta_i \) as in (9).

As a particular case, when taking \( \phi_1 \) linear, we retrieve condition (10) from the SEU benchmark in Section 5. We know that in this special case, for specific pairs of conditions on \( \tilde{c}_t \) and \( u \), the inequality is met if \( -k''(U)/k'(U) \) is nonincreasing.

Observe, that according to the pricing formula (4) an increase in parameter \( \eta \) always decreases the socially efficient discount rate in our analytical example. However, the following example shows that in general, neither \( k \) concave, nor \( -k''(U)/k'(U) \) nonincreasing are enough to assure a clear-cut effect.

Consider a risk neutral agent and \( n = 2 \) plausible distribution functions \( F_1 \) and \( F_2 \) for future consumption with expectations \( \tilde{c}_{t1} = 1 \) or \( \tilde{c}_{t2} = 6 \). We assume that the these two distributions are equally likely to be the true one, i.e., \( q_1 = q_2 = 1/2 \). Further, we assume that the rate of pure preference for the present \( \delta \) equals zero. Suppose now that agent 1 has logarithmic ambiguity preferences \( \phi_1(U) = \ln U \) and that agent 2 is more ambiguity averse \( \phi_2(U) = k(\phi_1(U)) \) with the transformation function \( k(x) = -0.5 \exp(-2x) \). Note that \( k(x) \) is increasing, concave and that indeed \( -k''(U)/k'(U) \) is nonincreasing. However, tedious computations yield

\[
a_2 E u'(\tilde{c}_{t2}) = 0.70 \leq 1.43 = a_1 E u'(\tilde{c}_{t1}).
\]

We know from (??) that this is equivalent to an increase in the socially efficient discount rate \( r_{t1} \leq r_{t2} \).
In particular, the wealth effect in this example decreases, i.e. \( a_2 \leq a_1 \). In contrast, the following result shows that the increase in ambiguity aversion has a clear effect on the implicit pessimism of beliefs.

**Lemma 4** Suppose without loss of generality that \( \text{Eu}(\tilde{c}_{t1}) \leq \ldots \leq \text{Eu}(\tilde{c}_{tn}) \). The following two conditions are equivalent:

1. Beliefs \( \tilde{q}_2 \) are dominated by \( \tilde{q}_1 \) in the sense of the monotone likelihood ratio order, for any set of marginals \( (\tilde{c}_{t1},\ldots,\tilde{c}_{tn}) \) satisfying the above-mentioned ranking.

2. \( \phi_2 = k(\phi_1) \) is more ambiguity-averse than \( \phi_1 \), such that \( k \) is increasing and concave.

**Proof:** Note that we need to find that \( \tilde{q}_2/q_1 \) and \( \text{Eu}(\tilde{c}_{t}) \) are anti-comonotonic. Using (9), we can rewrite the ratio as

\[
\frac{\tilde{q}_2}{q_1} = k'(\phi_1(\text{Eu}(\tilde{c}_{t}))) \frac{\sum_{\tau=1}^{n} q_\tau \phi_1'(\text{Eu}(\tilde{c}_{\tau}))}{\sum_{\tau=1}^{n} q_\tau \phi_2'(\text{Eu}(\tilde{c}_{\tau}))}.
\]

Note, that the fraction on the right hand side does not change with \( \theta \). Furthermore, \( k' \) is decreasing in its argument. Finally, since by assumption, the argument \( \phi_1(\text{Eu}(\tilde{c}_{t}))) \) is itself increasing with \( \theta \), we get the desired result. \( \blacksquare \)

Hence, in order to find a clear-cut effect on the equilibrium rate, one needs to determine whether the change in the wealth effect offsets the impact of increased pessimism. For instance, the wealth effect in our analytical example in Section 3 is linearly increasing in \( \eta \), such that the effect of an increase in \( \eta \) is unambiguously negative. We cannot provide sufficient conditions which guarantee a clear effect on \( r_t \) in general. Instead, the remainder focuses on the CAAA family suggested by Klibanoff, Marinacci and Mukerji (2005).

**Lemma 5** Suppose that in economy 1 we have constant absolute ambiguity aversion \( \phi_1 = -A^{-1} \exp(-AU) \). Then in economy 2 with \( \phi_2 = k(\phi_1) \) such that \( k \) is increasing, concave and \( (-k''(x)/k'(U))' \leq 0 \) for all \( x \), the discount rate is lower if \( (\text{Eu}(\tilde{c}_{t}))_{g=1,\ldots,n} \) and \( (\text{Eu}'(\tilde{c}_{t}))_{g=1,\ldots,n} \) are anti-comonotonic.

**Proof:** From Lemma 4, we know that it is sufficient to show that the wealth effect will increase, i.e. \( a_j \geq a_i \). Note that due to properties of the
exponential function $a_i = 1$ since $-A\phi_i(U) = \phi_i'(U)$. Moreover, the fact that $(-k''(x)/k'(U))' \leq 0$ together with $\phi_i$ exhibiting CAAA, yields that $\phi_2$ is DAAA. Finally, we know from Lemma 2, that for any DAAA function, the wealth effect $a_j \geq 1 = a_i$. ■

Given that the wealth effect vanishes in a CAAA economy, we know that introducing DAAA and at the same time increasing ambiguity aversion increases marginal expected utilities. Hence, the discount rate should decrease. Obviously, Lemma 5 allows for an increase in the CAAA parameter $A$, i.e. a $k$ which preserves CAAA such that $A_2 > A_1$.

Thanks to the results on anti-comonotonicity in the previous section we are now able to generalize the sufficient conditions beyond the SEU case.

**Proposition 5** Suppose that in economy $i$ we have constant absolute ambiguity aversion $\phi_i = -A^{-1}\exp(-AU)$. Then in any economy $j$ with $\phi_j = k(\phi_i)$ such that $k$ is increasing, concave and $(-k''(x)/k'(U))' \leq 0$ for all $x$, the discount rate is lower if one of the following conditions is satisfied:

1. The set of marginals $(\tilde{c}_{t1},...\tilde{c}_{tn})$ can be ranked according to FSD and $u$ is increasing and concave.

2. The set of marginals $(\tilde{c}_{t1},...\tilde{c}_{tn})$ can be ranked according to SSD and $u$ is increasing, concave and exhibits prudence.

3. The set of marginals $(\tilde{c}_{t1},...\tilde{c}_{tn})$ can be ranked according to Jewitt (1989) and $u$ is increasing and concave and exhibits DARA.

Thus, take any CAAA economy and suppose that agents rank scenarios according to conditions 1., 2. or 3. Then Proposition 5 tells us that future costs should be discounted at a lower rate in any more ambiguity averse DAAA economy. In such a case, we can guarantee that the wealth effect does not offset the increased pessimism due to more ambiguity aversion.

7 Conclusion

As indicated by recent literature, parameter uncertainty might well be decisive in evaluating costs and benefits across a long time-range. This paper provides comparative statics results when introducing an ambiguity-averse representative agent. In general, we cannot say that ambiguity aversion shifts
the socially efficient discount down. However, we can identify sufficient conditions for a clear-cut effect. Under DAAA and moderate requirements on risk-attitudes and the ranking of plausible consumption scenarios, cash-flows should in effect be discounted at a smaller rate. Moreover, for the CAAA family which include the maxmin expected utility functions from Gilboa and Schmeidler (1989), we state conditions which guarantee an unambiguous effect of being more ambiguity averse on the discount rate.

Unfortunately, to our knowledge, there exists no empirical literature which aims at specifying smooth ambiguity preferences yet. We might conjecture that non increasing absolute ambiguity aversion is plausible by the parallels to its well-established cousin in risk-theory, DARA. But this is all the more unsatisfactory since our model suggests that the properties of ambiguity attitude might be decisive for long-term cost-benefit analyses.

Clearly, this paper abstracts from several important aspects of a dynamic economy. In particular, we do not explicitly model the evolution of growth. Hence, a natural extension would characterize the term structure under ambiguity aversion more generally than our analytical example. In addition, a model which allows for learning could resume to investigate the optimal timing of long-term investments with ambiguous benefits.
References


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Appendix: Proof of Equation (4)

This proof relies on the well-known property that the Arrow-Pratt approximation is exact with an exponential function and a normally distributed random variable. For example, this property implies that

\[
Eu(\tilde{c}_{t\theta}) = (1 - \gamma)^{-1} E \exp \left[ (1 - \gamma)(\ln \tilde{c}_{t\theta}) \right] = (1 - \gamma)^{-1} \exp \left[ (1 - \gamma)(\ln c_0 + \theta t + 0.5(1 - \gamma)\sigma^2 t) \right].
\]

Similarly, we have that

\[
Eu'(\tilde{c}_{t\theta}) = \exp \left[ -\gamma(\ln c_0 + \theta t - 0.5\gamma\sigma^2 t) \right].
\]

Because \(\tilde{\theta}\) is normally distributed, we obtain that

\[
\phi(V_t(0)) = E\phi(Eu(\tilde{c}_{t\theta})) = h k (1 - \gamma)^{-1+\frac{2}{k}} E \exp \left[ (1 - \gamma)(1 - \frac{\eta}{k}) \times (\ln c_0 + \tilde{\theta} t + 0.5(1 - \gamma)\sigma^2 t) \right] \times \left(1 - \frac{\gamma}{k}\right)^{1+\frac{2}{k}} \exp \left[ (1 - \gamma)(1 - \frac{\eta}{k}) \times (\ln c_0 + \tilde{\theta} t + 0.5(1 - \gamma)\sigma^2 t + 0.5(1 - \gamma)(1 - \eta/k)\sigma^2 t^2) \right],
\]

which implies in turn that

\[
\phi'(V_t(0)) = h k \left(1 - \gamma\right)^{\frac{2}{k}} \exp \left[ -(1 - \gamma)\frac{\eta}{k} \times (\ln c_0 + \mu t + 0.5(1 - \gamma)\sigma^2 t + 0.5(1 - \gamma)(1 - \eta/k)\sigma^2 t^2) \right].
\]

We also obtain that

\[
\phi'(Eu(\tilde{c}_{t\theta})) = h k \left(1 - \gamma\right)^{\frac{2}{k}} \exp \left[ -\frac{\eta}{k} (1 - \gamma)(\ln c_0 + \tilde{\theta} t + 0.5(1 - \gamma)\sigma^2 t) \right],
\]

so that

\[
E\phi'(Eu(\tilde{c}_{t\theta})) Eu'(\tilde{c}_{t\theta}) = h k \left(1 - \gamma\right)^{\frac{2}{k}} E \exp \left[ \frac{\eta(\gamma - 1)}{k} - \gamma \right] (\ln c_0 + \tilde{\theta} t) + 0.5\sigma^2 t \left(\gamma^2 - \frac{\eta(1 - \gamma)^2}{k}\right).
\]
It implies that

\[ E\phi'(Eu(\tilde{c}_{t\theta}))Eu'(\tilde{c}_{t\theta}) = \text{hk} \left( \frac{1 - \gamma}{k} \right)^{\frac{k}{2}} \exp \left[ \left( \frac{\eta(\gamma - 1)}{k} - \gamma \right) \times \right. \]

\[ \times \left( \ln c_0 + \mu t + 0.5\sigma_0^2 t^2 \left( \frac{\eta(\gamma - 1)}{k} - \gamma \right) + \right. \]

\[ \left. + 0.5\sigma^2 t \left( \gamma^2 - \eta \left( 1 - \frac{\gamma}{k} \right) \right) \right) \]

All this implies that

\[ \ln \frac{E\phi'(Eu(\tilde{c}_{t\theta}))Eu(\tilde{c}_{t\theta})}{u'(c_0)\phi'(V_t(0))} = -\mu\gamma t + 0.5\gamma^2\sigma^2 t + 0.5\sigma_0^2 t^2 \left( \gamma^2 \left( 1 - \frac{\eta}{k} \right) + \frac{\eta}{k} \right). \]

By equation (3), we immediately get equation (4). $\blacksquare$